

An application of CAT

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Abstract

We comment on a question of Justin Moore on colourings of pairs of nodes in an Aronszajn tree.

Lemma 1.1. For any uncountable antichain A in an Aronszajn tree T , there is an infinite chain C in T such that every element of C is the meet of two elements of A .

Proof. Define a sequence $\langle (t_i, u_i, v_i, B_i) : i < \omega \rangle$ by recursion as follows. To start let t_0 be an arbitrary element of A (note that t_0 cannot be the root of T). Consider all $t_0 \wedge a$ for $a \in A$ and notice that this is a countable set. Hence there is $u_0 <_T t_0$ such that $\{a \in A : u_0 = a \wedge t_0\}$ is uncountable. Note that u_0 has at least two distinct immediate successors in T , so let v_0 be an immediate successor of u_0 which is incompatible with t_0 and which has uncountably many extensions in A . Denote the set of extensions of v_0 in A by B_0 .

At the stage $i = j + 1$ we choose $t_i \in B_j$ and $u_i <_T t_i$ such that $\{a \in B_j : u_i = a \wedge t_i\}$ is uncountable. Choose v_i to be an immediate successor of u_i which is incompatible with t_i and which has uncountably many extensions in B_j . Let this set of extensions be B_i . Note that $u_j <_T v_j <_T u_i <_T t_i$.

At the end the set $\{u_i : i < \omega\}$ is an infinite chain such that $u_i = t_i \wedge t_{i+1}$. ★_{1.1}

Definition 1.2. (1) A *subtree* of an ω_1 -tree T will mean an uncountable meet closed subset of T .

(2) A subtree S of an ω_1 -tree is *binary* if every node of S has at most two distinct immediate successors in S .

(3) An ω_1 -tree T is *binarisable* if every subtree of T has a binary subtree.

(4) For a tree T we let $T^{[2]} = \{\{s, t\} : s <_T t\}$.

The above definition will be used in the context of Aronszajn trees, so the case of trivial binary trees, namely uncountable branches, will be avoided. We will use the following statement introduced in [1] and used in [3]:

Colouring Axiom for Trees (CAT): For any partition $T = K_0 \cup K_1$ of an Aronszajn tree T , there is an uncountable set $X \subseteq T$ and $i < 2$ such that $x \wedge y \in K_i$ for all distinct $x, y \in X$.

We remark that by repeated applications of CAT one obtains that for any partition of an Aronszajn tree T into finitely many pieces there is an uncountable set $X \subseteq T$ such that $x \wedge y$ lie in the same piece of the partition, for all distinct $x, y \in X$.

Theorem 1.3. (CAT) For every binarisable special Aronszajn tree $T \subseteq {}^{\omega_1}>p$ there is a colouring c of $T^{[2]}$ into $p + 1$ colours such that every subtree S of T realises at least 3 colours.

Proof. Let $T \subseteq {}^{\omega_1}>p$ be a given binarisable special Aronszajn tree. Let $T = \bigcup_{n < \omega} A_n$ witness that T is special, so each A_n is an antichain. We shall assume that A_n 's are disjoint, and for $t \in T$ let $n(t)$ be n such that $t \in A_n$. Note that if $s <_T t$ then $n(s) \neq n(t)$. Define $c : T^{[2]} \rightarrow p + 1$ for $s <_T t$ by:

$$c(\{s, t\}) = \begin{cases} p & \text{if } n(s) < n(t), \\ i & \text{if } n(s) > n(t) \text{ and } t(\lg(s)) = i. \end{cases}$$

Here $\lg(s)$ is the order type of the domain of s . Let now S be any subtree of T . Let $S' \subseteq S$ be a binary subtree of S . Order the subsets of p by $<^*$. Let $d : S' \rightarrow \mathcal{P}(p)$ be given by

$$d(s) = \{j : s \frown j \text{ has an extension in } S'\}.$$

Applying CAT we obtain an uncountable $X \subseteq S'$ and $J \subseteq p$ such that for all $x \neq y \in X$ we have $d(x \wedge y) = J$. Since S' is binary the cardinality of J is at most 2, but since X is uncountable and T has no uncountable branches, J must have cardinality exactly 2. Let S'' be the set of meets of distinct elements of X and note that this is a subtree of S' , hence of S . Also, every node of S'' has the property that $s \frown j$ has an extension in X iff $j \in J$.

To complete the proof we shall show that for all $j \in J \cup \{p\}$ there is $\{s, t\} \in S''^{[2]}$ such that $c(\{s, t\}) = j$. Let m be such that $S'' \cap A_m$ is uncountable and apply Lemma 1.1 to $S'' \cap A_m$ to obtain an infinite chain $C \subseteq S''$ such that

$$(\forall x \in C)(\exists y, z \in S'' \cap A_m) x = y \wedge z.$$

Note that $s \neq t \in C$ implies $n(s) \neq n(t)$, hence there are s_p and t_p in C such that $c(\{s_p, t_p\}) = p$. Let $x \in C$ be such that $n(x) > m$. Find $y \neq z \in S'' \cap A_m$ such that $x = y \wedge z$. Since $x \in S''$, $d(x) = J$. Hence $\{c(\{x, y\}), c(\{x, z\})\} = J$. ★_{1.3}

With $p = 2$ Theorem 1.3 states that under CAT, for every special Aronszajn subtree of $\subseteq {}^{\omega_1}>2$, there is a colouring of $T^{[2]}$ into 3 colours such that every subtree S of T realises all 3 colours. Since CAT is implied by PFA, which also implies that all Aronszajn trees are special, this answers negatively Question 9.6. of [2]. For further references and results on CAT see [2], especially Theorem 5.2.

References

- [1] U. Abraham and S. Shelah, *Isomorphism types of Aronszajn trees*, Israel Journal of Mathematics, 50 (1-2): 75-113, 1985.
- [2] J.T. Moore, *Structural Analysis of Aronszajn trees*, preprint to appear in the Proceedings of the 2005 Logic Colloquium in Athens, Greece.
- [3] S. Todorcevic, *Lipschitz maps on trees*, report 2000/01 number 13, Mittag-Leffler Institute.